

COMMUTING GRAPHS ON FINITE SUBGROUPS OF $SL(2, \mathbb{C})$ AND DYNKIN DIAGRAMS

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ABSTRACT. Finite subgroups of $H \subset SL(2, \mathbb{C})$ and their relation with the so called ADE classification have played a central role in several areas of mathematics. In this paper we study commuting graphs $\mathcal{C}(H, \Gamma)$ for every finite subgroup $H \subset SL(2, \mathbb{C})$ for different subsets $\Gamma \subseteq H$, and investigate metric properties of them. Moreover, we realise any simple graph as a commuting graph of a certain Coxeter group, so in particular we recover every Dynkin diagram of ADE type as a commuting graph.

1. INTRODUCTION

Finite subgroups of $H \subset SL(2, \mathbb{C})$ were classified by F. Klein around 1870 into the families of cyclic C_n , binary dihedral groups BD_{4n} , and the exceptional cases of binary tetrahedral BT_{24} , binary octahedral BO_{48} and binary icosahedral BI_{120} . From the study of the (orbifold) quotients \mathbb{C}^2/H , also known as Du Val singularities, these groups have played a central role in many areas of algebraic geometry, singularity theory, simple Lie algebras, representation theory, McKay correspondence, and group theory. See [7], [12], [14], [15] for some references and further reading.

For a group H and a non empty subset $\Gamma \subseteq H$, the commuting graph $G = \mathcal{C}(H, \Gamma)$ is the graph with Γ as the node set and where any $x, y \in \Gamma$ are joined by an edge if x and y commute in H . This graph have been considered by several authors in various contexts like [1], [6] for groups of matrices, [3] for symmetric groups and [2], [16] for dihedral groups. In this paper we study commuting graphs for every subgroup $H \subset SL(2, \mathbb{C})$ and in the spirit of [2] we discuss distance and detour distance properties, metric dimension and the resolving polynomial in each case.

One of the most remarkable links among topics related to finite subgroups of $SL(2, \mathbb{C})$ is the so called *ADE classification* or simple laced Dynkin diagrams. Indeed, they classify at the same time finite subgroups of $SL(2, \mathbb{C})$, Du Val singularities, simple Lie algebras, (simple laced) finite Coxeter groups, quivers of finite type among others. By considering commuting graphs on Coxeter groups, we are able to realise any simple graph as a commuting graph of a certain Coxeter group W , so we conclude the paper by recovering the ADE classification as commuting graphs $\mathcal{C}(W, \Gamma)$, taking Γ as the set of generators of W .

The paper is distributed as follows: Section 2 introduces the basic notations and results needed. Section 3 studies the cyclic case C_n , which is trivial in this set up. Section 4 studies the binary dihedral BD_{4n} which contains the main calculations for the metric properties. Sections 5, 6 and 7 study the exceptional cases BT_{24} , BO_{48} and BI_{120} using

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the software GAP [8], and finally Section 8 recovers simple graphs as commuting graphs and the particular case of Dynkin diagrams.

2. BASIC NOTIONS

A graph G is a set of nodes $V(G)$ and edges $E(G)$. The total number of nodes of the graph G , denoted by $|G|$, is called the order of G . Two nodes a and b are called adjacent if there is an edge between them, and we denote them by $a \sim b$. Otherwise we write $a \not\sim b$. For definitions and further explanations see [4] and [5].

Let a, b be two nodes in a graph G . The distance from a to b , denoted by $d(a, b)$, is the length of a shortest path between a to b in G . Also the length of a longest path from a to b in G is denoted by $d_D(a, b)$. An $a - b$ path of length $(d_D(a, b))$ $d(a, b)$ is known as $a - b$ (detour) geodesic respectively.

The largest distance between a node a and any other node of G is called (detour) eccentricity, denoted by $(e_D(a))$ $e(a)$. The (detour) diameter $(d_D(G))$ $d(G)$ of the graph G , is the greatest (detour) eccentricity among all the nodes of the graph G . Also the (detour) radius $(r_D(G))$ $r(G)$ of the graph G , is the smallest (detour) eccentricity among all the nodes of the graph G . For metric properties of graphs, see for example [4], [5], [9] and [10].

A node y is said to be an eccentric node for x if $e(x) = d(x, y)$. A node z is an eccentric node of G , if z is an eccentric node of certain nodes of the graph G . The eccentric subgraph is denoted by $Ecc(G)$, is a graph induced by the eccentric nodes of the graph G . If all the nodes of the graph G are eccentric nodes of G , then G is called eccentric graph. The closure of G is denoted by $Cl(G)$ is the graph achieve from G repeatedly joining edges non adjacent nodes until no such edge remains of G whose degrees sum is at least m , where m is the order of G . If $Cl(G) = G$ then G is called closed graph, see for further details [2] and [13].

A node a is known as (detour) peripheral node in a graph G , if $(e_D(a) = d_D(G))$ $e(a) = d(G)$, and a node a is known as (detour) central node of the graph G , if $(e_D(a) = r_D(G))$ $e(a) = r(G)$. Subgraphs induced by the (detour) peripheral nodes and (detour) central nodes of the graph G , are called (detour) periphery and (detour) center respectively, these are denoted by $(Pe_D(G))$ $Pe(G)$ and $(Ce_D(G))$ $Ce(G)$ respectively.

The degree $deg(z)$ of a node z in a graph G is the number of nodes in G that are incident to z where the loops are counted twice. The detour degree $deg_D(z)$ of z is the total number of $D(z)$, where $D(z) = \{y \in V(G) : d_D(y, z) = e_D(z)\}$. The average (detour) degree of the graph G , is the quotient of the total number of (detour) degrees of $V(G)$ and the order of the graph G given by $(Deg_{av}(G))$ $deg_{av}(G)$. The detour degree sequence $(Deg(G))$ $deg(G)$ is the non increasing sequence of nodes of the graph G . For $z \in V(G)$ if $(Deg_j(z))$ $deg_j(z)$ is the list of nodes at (detour) distance j from z , then $((Deg_0(z), Deg_1(z), Deg_2(z), \dots, Deg_{e_D(z)}(z)))$ $(deg_0(z), deg_1(z), deg_2(z), \dots, deg_{e_D(z)}(z))$ known as (detour) distance degree sequence of z and is denoted by $(\mathcal{D}_D(z))$ $\mathcal{D}(z)$. So we get the sequel remark useful in Section 4.

- Remark 2.1.** (1) *The length of $(\mathcal{D}_D(x))$ $\mathcal{D}(x)$ sequence is one more than $e_D(x)$.*
 (2) *Clearly $Deg_0(x) = deg_0(x) = 1$, $deg_1(x) = deg(x)$ and $Deg_{e_D}(x) = deg_D(x)$.*
 (3) $\sum_{j=0}^{e_D(x)} deg_j(x) = \sum_{j=0}^{e_D(x)} Deg_j(x) = |G|$.

In a graph G , the (detour) distance degree sequence $(\mathcal{D}_D(G))$ $\mathcal{D}(G)$ is the set of $(\mathcal{D}_D(x))$ $\mathcal{D}(x)$ sequences of $x \in V(G)$. In this paper, the sequence (q^l, r^m, s^n) denotes q appear l times, r appear m times and s appear n times.

Let $u \in G$ be a node. The neighbourhood of u , is the set of all nodes of G incident to u , we denote it by $N(u)$. A node x of G is called an interior node of G if any $u \in V(G)$, distinct from the node x , there exist v such that x lies between u and v at the same distance. Interior of G is denoted by $Int(G)$ is a subgraph induced by the interior nodes of the graph G . For $x \in V(G)$, if the subgraph induced by the neighbourhood of x is complete, the x is a complete node. A node x of G is called boundary node if $d(x, y) \leq d(x, z)$ for every neighbour y of z , although a node x is called a boundary node of the graph G if x is boundary node of some nodes of a graph G , see [2] and [13] for further details. The following theorems will be used in Section 4.

Theorem 2.2. [4] *A graph G is connected and $x \in V(G)$, then x is a boundary node of each node other than x iff x is complete node of the graph G .*

Theorem 2.3. [4] *A graph G is connected and $x \in V(G)$, then x is a boundary node of the graph G iff $x \notin Int(G)$.*

The open neighbourhood of $y \in V(G)$ is defined as $N(y) = \{z \in V(G) : y \sim z \text{ in } G\}$, also the closed neighbourhood of y is defined as $N[y] = N(y) \cup \{y\}$. For any $x, y \in G$ such that $x \neq y$, if $N[x] = N[y]$ then x and y are called adjacent twins, if $N(x) = N(y)$ then x and y are called non adjacent twins. We mention that if x, y commute in G then x, y are adjacent twins, and if not then x, y are non adjacent twins. Furthermore if x, y are adjacent twins or non adjacent twins, then x, y are called twins. Any subset W of $V(G)$ is said to be a twin set in a graph G if x, y are twins in a graph G for each pair of distinct x, y nodes of W .

For an ordered subset $U = \{u_1, u_2, \dots, u_m\}$ of the node set $V(G)$ and a node $t \in G$, an m -vector $s(t|U) = (d(t, u_1), d(t, u_2), \dots, d(t, u_m))$ is said to be the representation of t with respect to U . A set U is said to be a locating set (or resolving set) for G if any two different nodes of the graph G have different representation with respect to U . Now, a basis of G is a minimum locating set for the graph G , and the metric dimension $\psi(G)$ is the cardinality of a basis of G , see [5] and [9] for further details.

A j -subset is a subset of $V(G)$ of order j . Let $\mathcal{R}(G, j)$ be the family of locating sets for G which are j -subsets and suppose let $s_j = |\mathcal{R}(G, j)|$. Then we define the resolving polynomial $\psi(G, x)$ of the graph G , as $\psi(G, x) = \sum_{j=\psi(G)}^m s_j x^j$, where m is the order of G and $\psi(G)$ is the metric dimension of G . It is clear that $s_j = 0$ if and only if $m < j$ or $j < \psi(G)$. The resolving sequence is the coefficients of $\psi(G, x)$. We recall the following results that will be used in Section 4.

Lemma 2.4. [9] *If a graph G is connected and x, y are twins in G , then $d(x, z) = d(y, z)$ for each node $z \in V(G) \setminus \{x, y\}$.*

Corollary 2.5. [9] *If a graph G is connected and x, y are twins in G , then x or y is in W . Furthermore, if $x \in W$ and $y \notin W$, then $(W \setminus \{x\}) \cup \{y\}$ also resolves G .*

Remark 2.6. *Let G be a connected graph, m is the order of G and W is a twins set in G with $|W| = k \geq 2$, then each resolving set for a graph G contain at least $k - 1$ nodes of W .*

Let G_1 and G_2 be any two graphs. The join of G_1 and G_2 , denoted by $G_1 \vee G_2$, is the graph with node set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{y \sim z : y \in V(G_1), z \in V(G_2)\}$. A graph K_n with n nodes is called complete graph if any two different nodes of K_n are connected by exactly one edge.

3. CICLIC SUBGROUPS C_n

Since $C_n = Z(C_n)$ this case is trivial. For any subset $\Gamma \subseteq C_n$ we have that $G = \mathcal{C}(C_n, \Gamma) = K_n$ the complete graph with $n = |\Gamma|$, so all the metric properties of G are obvious.

4. BINARY DIHEDRAL SUBGROUPS BD_{4n}

The presentation of binary dihedral group BD_{4n} of order $4n$ ($n \geq 2$) is given by

$$BD_{4n} = \langle \alpha, \beta : \alpha^{2n} = \beta^4 = 1, \alpha^n = \beta^2, \alpha\beta = \beta\alpha^{-1} \rangle.$$

The center of binary dihedral group is $Z(BD_{4n}) = \{e, \alpha^n\}$ and consider the following subsets of BD_{4n} :

$$\begin{aligned}\Gamma_1 &= \{1, \alpha, \alpha^2, \dots, \alpha^{2n-1}\}, \\ \Gamma_2 &= \{\beta, \alpha\beta, \alpha^2\beta, \dots, \alpha^{2n-1}\beta\}, \\ \Gamma_3 &= \Gamma_1 \setminus Z(BD_{4n}).\end{aligned}$$

Observe that $\Gamma_2 = \bigcup_{j=0}^{n-1} \Gamma_2^j$, where $\Gamma_2^j = \{\alpha^j\beta, \alpha^{n+j}\beta\}$.

We collect in the following Lemma some basic results from [2] that will be used in the rest of the paper.

Lemma 4.1 ([2], 1.1, 1.2, 1.3, 1.4). *Let $G = \mathcal{C}(H, H)$ be the commuting graph on a group H of order n . Then*

- (1) $r(G) = 1$.
- (2) $d(G) = \begin{cases} 1 & , \text{ if } H \text{ is abelian,} \\ 2 & , \text{ if } H \text{ is non-abelian.} \end{cases}$
- (3) $Ce(G)$ is induced by $Z(H)$.
- (4) $G = \begin{cases} K_n & , \text{ if } H \text{ is abelian,} \\ Ce(G) \vee Pe(G) & , \text{ if } H \text{ is non-abelian.} \end{cases}$
- (5) $Pe(G)$ is induced by $H \setminus Z(H)$.

Let $y = \alpha^i\beta$ and $z = \alpha^{n+i}\beta$, where $0 \leq i \leq n-1$ are two elements of Γ_2 . It is easy to check that y and z commute in BD_{4n} , but $yz \neq zy$ when $y \in \Gamma_2$ and $z \in \Gamma_3$. Thus we have the following proposition.

Proposition 4.2. *Let BD_{4n} be a binary dihedral group and let $G = \mathcal{C}(BD_{4n}, \Gamma)$ be a commuting graph on BD_{4n} . We have*

$$G \cong \begin{cases} K_2 & , \text{ when } \Gamma = Z(BD_{4n}), \\ nK_2 & , \text{ when } \Gamma = \Gamma_2, \\ K_{2n-2} & , \text{ when } \Gamma = \Gamma_3, \end{cases}$$

where nK_2 denotes n copies of the complete graph K_2 .

Proof. If $\Gamma = Z(BD_{4n})$, but $Z(BD_{4n}) = \{e, \alpha^n\}$, both elements commute with each other in BD_{4n} , so it is complete and is denoted by K_2 .

If $\Gamma = \Gamma_2$, let $x = \alpha^i\beta$ and $y = \alpha^{n+i}\beta$, where $0 \leq i \leq n-1$ are two elements of Γ_2 we get $xy = yx$, thus G is n copies of K_2 . Finally if $\Gamma = \Gamma_3$, then any two distinct elements of Γ_3 commute in BD_{4n} , so it is complete and denoted by K_{2n-2} . \square

Since $BD_{4n} \setminus Z(BD_{4n}) = \Gamma_2 \cup \Gamma_3$, as we know order of Γ_2 is $2n$ and Γ_3 is $2(n-1)$. We have the following result, that can be prove by using above results.

Proposition 4.3. *Let $H = BD_{4n}$ be a binary dihedral group and let $G = \mathcal{C}(H, H)$ be a commuting graph. Then*

$$G = K_2 \vee (K_{2n-2} \cup nK_2).$$

4.1. Distant and detour distant properties.

Theorem 4.4. *Let $H = BD_{4n}$ for $n \geq 2$ be a binary dihedral group and let $G = \mathcal{C}(H, H)$. We have*

(1)

$$e_D(x) = \begin{cases} 2n+1 & , \text{for all } x \in Z(BD_{4n}) \\ 2n+2 & , \text{for all } x \in \Gamma_2 \cup \Gamma_3, \end{cases}$$

(2) $r_D(G) = 2n+1$,

(3) $d_D(G) = 2(n+1)$.

Proof. (1) Since every $x \in Z(BD_{4n})$ commutes with all elements of BD_{4n} . If $x_1, y_1 \in \Gamma_3$ then $x_1 y_1 = y_1 x_1$ in BD_{4n} . Also elements of the form x^i, y^i in Γ_2^i , where $0 \leq i \leq n-1$, satisfies $x^i y^j = y^j x^i$. If $u \in \Gamma_2$ and $v \in \Gamma_3$, then $uv \neq vu$. Thus we get

- (i) For every $x \in Z(BD_{4n})$ there is a $x - z$ path of detour length $2n+1 \forall z \in Z(BD_{4n}) \setminus \{x\}$; an $x - z$ path of detour length $2n+1 \forall z \in \Gamma_2 \cup \Gamma_3$.
- (ii) For every $0 \leq i \leq n-1, y \in \Gamma_2^i$ there is a $y - z$ path of detour length $2n+1 \forall z \in \Gamma_2^i \setminus \{y\}$ and for every $z \in \Gamma_2^j$ s.t $i \neq j$. Also there is $y - z$ path of detour length $2n+2, \forall z \in \Gamma_3$. Finally
- (iii) For every $y \in \Gamma_3$, a $y - z$ path of detour length $2n+1 \forall z \in \Gamma_3 \setminus \{y\}$.

The proofs of (2) and (3) are straight forward and follows from (1). □

Remark 4.5. $Pe_D(G)$ is induced by $BD_{4n} \setminus Z(BD_{4n})$, and so $Pe_D(G) \simeq Pe(G)$. Also $Ce_D(G)$ is induced by $Z(BD_{4n})$, and so $Ce_D(G) \simeq Ce(G)$.

Theorem 4.6. *Let $H = BD_{4n}$ be a binary dihedral group and $G = \mathcal{C}(H, H)$. Then*

$$\mathcal{D}(G) = \{((1, 4n-1)^2, (1, 2n-1, 2n)^{2n-2}, (1, 3, 4n-4)^{2n})\},$$

and

$$\mathcal{D}_D(G) = \{((1, 0^2, 2n-2, 0^{2n-4}, 2n)^2, (1, 0^{2n}, 2n-1, 2n)^{2n-2}, (1, 0^{2n}, 2, 4n-3)^{2n})\}.$$

Proof. From Proposition 4.3, $G = K_2 \vee (K_{2n-2} \cup nK_2)$. For $x \in V(K_2)$, we get $e(x) = 1$ and by using Theorem 4.4 $e_D(x) = 2n+1$. So $\mathcal{D}(x) = (1, 4n-1)$ and $\mathcal{D}_D(x) = (1, 0, 0, 2n-2, \underbrace{0, 0, \dots, 0}_{2n-4 \text{ times}}, 2n)$. For $x \in V(K_{2n-2} \cup nK_2)$, we get $e(x) = 2$ and

again by using Theorem 4.4, $e_D(x) = 2n+2$. Therefore

$$\mathcal{D}(x) = \begin{cases} (1, 2n-1, 2n) & ; \forall x \in V(K_{2n-2}), \\ (1, 3, 4n-4) & ; \forall x \in V(nK_2), \end{cases}$$

and

$$\mathcal{D}_D(x) = \begin{cases} (1, \underbrace{0, 0, \dots, 0}_{2n \text{ times}}, 2n-1, 2n) & ; \forall x \in V(K_{2n-2}), \\ (1, \underbrace{0, 0, \dots, 0}_{2n \text{ times}}, 2, 4n-3) & ; \forall x \in V(nK_2). \end{cases}$$

As the cardinality of $V(K_2)$, $V(K_{2n-2})$, and $V(nK_2)$ are 2, $2(n-1)$, and $2n$ respectively. Hence $\mathcal{D}(G) = \{((1, 4n-1)^2, (1, 2n-1, 2n)^{2n-2}, (1, 3, 4n-4)^{2n})\}$, and $\mathcal{D}_D(G) = \{((1, 0^2, 2n-2, 0^{2n-4}, 2n)^2, (1, 0^{2n}, 2n-1, 2n)^{2n-2}, (1, 0^{2n}, 2, 4n-3)^{2n})\}$. \square

From Remark 2.1 following result follows.

Corollary 4.7. *Let BD_{4n} for $n \geq 2$ be the binary dihedral group and $G = \mathcal{C}(H, H)$ be the commuting graph on BD_{4n} . We have*

- (1) $\deg(G) = ((4n-1)^2, (2n-1)^{2n-2}, (3)^{2n})$,
- (2) $\text{Deg}(G) = ((4n-3)^{2n}, (2n)^{2n})$,
- (3) $\deg_{av}(G) = n+2$,
- (4) $\text{Deg}_{av}(G) = \frac{3(2n-1)}{2}$.

Proof. We know $Z(\text{BD}_{4n}) = \{e, \alpha^n\}$, since every $x \in Z(\text{BD}_{4n})$ commute with all elements of BD_{4n} . So the degree of x is $(4n-1)$, but $Z(\text{BD}_{4n})$ has only two elements thus the total degree of the elements of $Z(\text{BD}_{4n})$ is $(4n-1)^2$.

For each $x \in \Gamma_3$, the degree of x is $(2n-1)$, but the cardinality of Γ_3 is $(2n-2)$, therefore the total degree of the elements of Γ_3 is $(2n-1)^{2n-2}$. Finally for each $y \in \Gamma_2$ the degree of y is only 3, but the cardinality of Γ_2 is $(2n)$ thus $(3)^{2n}$.

Similarly the detour degree of every $x \in \Gamma_1$ is $(4n-3)$, and the cardinality of Γ_1 is $(2n)$ hence $(4n-3)^{2n}$. Also the detour degree of each $y \in \Gamma_2$ is $2n$, and the cardinality of Γ_2 is $(2n)$ therefore $(2n)^{2n}$.

Finally, $\deg_{av}(G) = \frac{4n^2+8n}{4n} = n+2$, and $\text{Deg}_{av}(G) = \frac{6n-3}{2} = \frac{3(2n-1)}{2}$. \square

Theorem 4.8. *Let $H = \text{BD}_{4n}$ be a binary dihedral group and $G = \mathcal{C}(H, H)$ be a commuting graph on BD_{4n} . Then*

$$\text{Int}(G) = \text{Ce}(G)$$

Proof. From above results we need to find all such nodes of the graph G which are complete, so we find $N(x) \forall x \in V(G)$. Since we know by Proposition 4.3, $G = K_2 \vee (K_{2n-2} \cup nK_2)$ so we

$$N(x) = \begin{cases} \Gamma_1 \setminus \{x\} & , \text{when } x \in V(K_{2n-2}), \\ \Gamma_2 \cup \{\Gamma_1 \setminus \{x\}\} & , \text{when } x \in V(K_2), \\ Z(\text{BD}_{4n}) \cup V((jK_2) \setminus \{x\}) & , \text{when } x \in jK_2, \text{ where } 1 \leq j \leq n. \end{cases}$$

Since, the subgraphs induced by $\Gamma_1 \setminus \{x\}$ and $Z(\text{BD}_{4n}) \cup V((jK_2) \setminus \{x\})$ are complete. This implies that every $x \in (K_{2n-2} \cup nK_2)$ is complete node, so by Theorem 2.2 x is a boundary node, and by using Theorem 2.3 $\Gamma_2 \cup \{\Gamma_1 \setminus \{x\}\}$ is not complete. Therefore every node of K_2 is an interior node of the graph G and $\text{Ce}(G) = K_2$. Hence we conclude that $\text{Int}(G) = \text{Ce}(G)$. \square

Theorem 4.9. *Let G be a commuting graph on BD_{4n} . Then*

$$\text{Ecc}(G) = G.$$

Proof. We know by Proposition 4.3, Corollary 4.4 and Theorem 2.2, that each x is an eccentric node of $(K_{2n-2} \cup nK_2)$. Next we show that every $x \in V(K_2)$ can be an eccentric node of some $y \in V(G) \setminus \{x\}$. As we know that $e(x) = 1 \forall x \in V(K_2)$ and $e(y) = 2 \forall y \in (K_{2n-2} \cup nK_2)$, thus $d(x, y) \neq e(y) \forall x \in V(K_2)$ and $\forall y \in (K_{2n-2} \cup nK_2)$. But $d(x, y) = e(y) \forall y \in V(K_2) \setminus \{x\}$. Therefore each $x \in V(K_2)$ is an eccentric node for the remaining nodes of K_2 . Thus we conclude that $\text{Ecc}(G) = G$. \square

Theorem 4.10. *Let G be a commuting graph on BD_{4n} . Then*

$$Cl(G) = G$$

Proof. Since we know $G = K_2 \vee (K_{2n-2} \cup nK_2)$. To prove the theorem first we find degree sum for every couple of nodes in the following cases.

- (i) The degree sums for every pair of nodes (y, z) of G s.t $y \in V(K_{2n-2})$, $z \in \bigcup_{j=1}^n V(K_2^j)$, and
- (ii) The degree sums for every pair of nodes (u, v) of G s.t $u \in V(K_2^i)$ and $v \in V(K_2^j)$, for $i \neq j$. Note that $\deg(y) = 2n - 1 \forall y \in V(K_{2n-2})$ also $\deg(z) = 3 \forall z \in V(K_2^j)$ for $0 \leq j \leq n$. Therefore $\deg(y) + \deg(z) = 2n + 2 < |G|$, and $\deg(u) + \deg(v) = 6 < |G|$. So $Cl(G) = G$. □

Lemma 4.11. *A connected graph G of order m is closed iff for each non adjacent couple of nodes (y, z) in G , we get $\deg(y) + \deg(z) < m$.*

Proof. Since we have $Cl(G) = G$ iff for each pair of non adjacent nodes (x, y) in G , we get $\deg(x) + \deg(y) < m$. Which is required. □

Corollary 4.12. *A graph $G = \mathcal{C}(H, H)$ on binary dihedral group BD_{4n} is closed.*

Proof. By Theorem 4.10 it is clear that G is closed. □

4.2. Resolving polynomial of commuting graphs on BD_{4n} .

Theorem 4.13. *Let BD_{4n} be a binary dihedral group, and $G = \mathcal{C}(H, H)$ commuting graph on BD_{4n} , we have $\psi(G) = \frac{3}{2}|\Gamma_2| - 2$.*

Proof. We know that $\Gamma_3, Z(BD_{4n})$ are both twins sets in G , the order of these sets are $2n - 2, 2$ respectively, next there are n twins sets Γ_2^i of order 2, where $0 \leq i \leq n - 1$. Thus by Remark 2.6, we get $\psi(G) \geq 3n - 2 = \frac{3}{2}(2n) - 2 = \frac{3}{2}|\Gamma_2| - 2$. Also it is obvious that $U = \{\alpha^i, \alpha^j\beta : 1 \leq i \leq 2n - 2, 0 \leq j \leq n - 1\}$ is the locating set for G of order $\frac{3}{2}|\Gamma_2| - 2$. Thus $\psi(G) \leq \frac{3}{2}|\Gamma_2| - 2$, therefore $\psi(G) = \frac{3}{2}|\Gamma_2| - 2$. □

The sequel result of resolving polynomial $\psi(G, x)$ of a commuting graph $G = \mathcal{C}(H, H)$ of order m is helpful.

Proposition 4.14. *Let G be a connected graph of order m . There is only one locating set of size m that is $V(G)$, and the locating set of size $m - 1$ for the graph G can be chosen m possible distinct ways, thus $s_{m-1} = m$ and $s_m = 1$, where s denotes the locating set.*

Proof. The proof is straight forward. □

Theorem 4.15. *Let G be a commuting graph on BD_{4n} , then we have*

$$\psi(G, x) = 2^{\frac{1}{2}|\Gamma_2|+1}|\Gamma_3|x^{\frac{3}{2}|\Gamma_2|-2} + |BD_{4n}|x^{|BD_{4n}|-1} + x^{|BD_{4n}|} + \sum_{j=\frac{3}{2}|\Gamma_2|-1}^{|BD_{4n}|-2} s_j x^j,$$

where $s_j = 2^{|BD_{4n}|-j-1}|\Gamma_3| \binom{\frac{1}{2}|\Gamma_2|+1}{|BD_{4n}|-j-1} + 2^{|BD_{4n}|-j} \binom{\frac{1}{2}|\Gamma_2|+1}{|BD_{4n}|-j}$.

Proof. By Theorem 4.13, $\psi(G) = \frac{3}{2}|\Gamma_2| - 2 = 3n - 2$. We find the resolving sequence $(s_{3n-2}, s_{3n-1}, \dots, s_{4n-1}, s_{4n})$ of length $n + 3$.

For s_{3n-2} , since $Z(\text{BD}_{4n})$, Γ_3 and Γ_2^j where $0 \leq j \leq n-1$ are twins sets. Thus by Corollary 2.5 and multiplication rule, we get

$$s_{3n-2} = \underbrace{\binom{2}{1}, \binom{2}{1}, \dots, \binom{2}{1}}_{n+1 \text{ times}} \binom{2n-2}{2n-3} = 2^{n+1}(2n-2).$$

For fixed j , s.t $3n-1 \leq j \leq 4n-2$, we find s_j . Let $u_1, u_2, u_3, \dots, u_{4n-j}$ be the nodes of G , s.t $u_1, u_2, u_3, \dots, u_{4n-j} \notin U$, where U is any locating set of order j . Then either $u_1, u_2, u_3, \dots, u_{4n-j} \in \bigcup_{j=0}^{n-1} \Gamma_2^j \cup Z(\text{BD}_{4n})$, or one of $u_1, u_2, u_3, \dots, u_{4n-j}$ say u_1 belongs to Γ_3 .

When $u_1, u_2, u_3, \dots, u_{4n-j} \in \bigcup_{j=0}^{n-1} \Gamma_2^j \cup Z(\text{BD}_{4n})$. As we know each Γ_2^j and $Z(\text{BD}_{4n})$ are twins sets of order 2, so by Remark 2.6, there is exactly one node of these $u_1, u_2, u_3, \dots, u_{4n-j}$ nodes belongs to exactly one set of the collection $\{Z(\text{BD}_{4n}), \Gamma_2^j : 0 \leq j \leq n-1\}$ of $n+1$ twins sets. Therefore by multiplication and addition, there are total

$$\binom{n+1}{4n-j} \binom{2}{1}^{4n-j} \binom{2n-2}{2n-2},$$

possible locating sets of order j s.t $u_1, u_2, u_3, \dots, u_{4n-j}$ does not belong to any locating set of order j .

Now, if one of $u_1, u_2, u_3, \dots, u_{4n-j}$, let say $u_1 \in \Gamma_3$, then by Remark 2.6, there is exactly one node form the nodes $u_1, u_2, u_3, \dots, u_{4n-j}$ of order $4n-j-1$ and belongs to just one set of the collection $\{Z(\text{BD}_{4n}), \Gamma_2^j : 0 \leq j \leq n-1\}$ of $n+1$ twins set. Therefore by multiplication and addition, there are total

$$\binom{n+1}{4n-j-1} \binom{2}{1}^{4n-j-1} \binom{2n-2}{2n-3},$$

possible locating sets of order j s.t $u_1, u_2, u_3, \dots, u_{4n-j}$ does not belong to any locating set of order j . Thus by addition rule,

$$s_j = 2^{4n-j-1}(2n-2) \binom{n+1}{4n-j-1} + 2^{4n-j} \binom{n+1}{4n-j}.$$

By Proposition 3.5, we get $s_{4n-1} = 4n$ and $s_{4n} = 1$. □

5. BINARY TETRAHEDRAL GROUP BT_{24}

Let the binary tetrahedral group be presented as

$$\text{BT}_{24} = \langle r, s, t \mid r^2 = s^3 = t^3 = rst \rangle$$

of order 24 and consider the following subsets of BT_{24} :

$$\begin{aligned} \mathcal{B}_1 &= Z(\text{BT}_{24}) = \{1, r^2\} & \mathcal{C}_1 &= \{s, s^{-1}, s^{-2}, tr\} \\ \mathcal{B}_2 &= \{r, r^{-1}\} & \mathcal{C}_2 &= \{rt, r^{-1}t, t^{-1}r^{-1}, t^{-1}r\} \\ \mathcal{B}_3 &= \{ts, s^{-1}t^{-1}\} & \mathcal{C}_3 &= \{t, t^{-1}, rs, s^{-1}r^{-1}\} \\ \mathcal{B}_4 &= \{tsr, s^{-1}t^{-1}r\} & \mathcal{C}_4 &= \{sr, t^{-1}s, s^{-1}t, r^{-1}s^{-1}\}. \end{aligned}$$

Proposition 5.1. *Let BT_{24} the binary tetrahedral group and $G = \mathcal{C}(\text{BT}_{24}, \Gamma)$ be a commuting graph on BT_{24} . Then we have*

$$G = \begin{cases} K_2 & , \text{when } \Gamma = \mathcal{B}_i \text{ for } i = 1, \dots, 4 \\ K_4 & , \text{when } \Gamma = \mathcal{C}_i \text{ for } i = 1, \dots, 4. \end{cases}$$

Since we obtain complete graphs, all the metric properties of G are obvious.

6. BINARY OCTAHEDRAL GROUP BO_{48}

Let the binary octahedral group be presented as

$$BO_{48} = \langle r, s, t \mid r^2 = s^3 = t^4 = rst \rangle.$$

Its order is 48 and consider the following subsets of BO_{48} :

$$\begin{aligned} \mathcal{B}_1 &= Z(BO_{48}) = \{1, r^2\} & \mathcal{C}_1 &= \{s, s^{-1}, s^{-2}, tr\} \\ \mathcal{B}_2 &= \{r, r^{-1}\} & \mathcal{C}_2 &= \{rt, r^{-1}t, t^{-1}r^{-1}, t^{-1}r\} \\ \mathcal{B}_3 &= \{ts, s^{-1}t^{-1}\} & \mathcal{C}_3 &= \{rts, r^{-1}ts, tsr, s^{-1}t^{-1}r\} \\ \mathcal{B}_4 &= \{s^{-1}ts^{-1}, st^{-1}s\} & \mathcal{C}_4 &= \{srt, t^2s, t^{-2}s, s^{-1}t^2\} \\ \mathcal{B}_5 &= \{t^{-1}r^{-1}t, t^{-1}rt\} & & \\ \mathcal{B}_6 &= \{r^{-1}tsr, rtsr\} & \mathcal{D}_1 &= \{t, t^2, t^{-1}, t^{-2}, rs, s^{-1}r^{-1}\} \\ \mathcal{B}_7 &= \{t^{-1}r^{-1}ts, t^{-1}rts\} & \mathcal{D}_2 &= \{sr, t^{-1}s, st^{-1}r^{-1}, st^{-1}r, s^{-1}t, r^{-1}s^{-1}\} \\ & & \mathcal{D}_3 &= \{t^2r, s^{-1}t^2s, srts, st^{-1}, ts^{-1}, srts\}. \end{aligned}$$

Proposition 6.1. *Let BO_{48} the binary octahedral group and $\mathcal{C}(BO_{48}, \Gamma)$ be a commuting graph on BO_{48} . Then we have*

$$G = \begin{cases} K_2 & , \text{when } \Gamma = \mathcal{B}_i \text{ for } i = 1, \dots, 7 \\ K_4 & , \text{when } \Gamma = \mathcal{C}_i \text{ for } i = 1, \dots, 4 \\ K_6 & , \text{when } \Gamma = \mathcal{D}_i \text{ for } i = 1, \dots, 3. \end{cases}$$

Since we obtain complete graphs, all the metric properties of G are obvious.

7. BINARY ICOSAHEDRAL GROUP BI_{120}

Let the binary icosahedral group be presented as

$$BI_{120} = \langle r, s, t \mid r^2 = s^3 = t^5 = rst \rangle.$$

The order of binary icosahedral group is 120 and consider the following subsets of BI_{120} :

$$\begin{aligned} \mathcal{B}_1 &= Z(BI_{120}) = \{1, r^2\} & \mathcal{B}_{15} &= \{rtsr, r^{-1}tsr\} \\ \mathcal{B}_2 &= \{r, r^{-1}\} & \mathcal{B}_{16} &= \{t^{-1}r^{-1}t, t^{-1}rt\} \\ \mathcal{B}_3 &= \{ts, s^{-1}t^{-1}\} & & \\ \mathcal{B}_4 &= \{s^{-1}ts^{-1}, st^{-1}s\} & \mathcal{C}_1 &= \{s, s^{-1}, s^{-2}, tr\} \\ \mathcal{B}_5 &= \{r^{-1}(tsr)^2, (rts)^2r\} & \mathcal{C}_2 &= \{rt, r^{-1}t, t^{-1}r^{-1}, t^{-1}r\} \\ \mathcal{B}_6 &= \{s^{-1}t^2srts, (srt)^2s\} & \mathcal{C}_3 &= \{s^{-1}t^{-2}, ts^{-1}t^{-1}, tst^{-1}, t^2s\} \\ \mathcal{B}_7 &= \{t^{-1}r^{-1}tsrts, t^{-1}(rts)^2\} & \mathcal{C}_4 &= \{t^{-1}rt^2s, s^{-1}t^2sr, srtsr, t^{-1}r^{-1}t^2s\} \\ \mathcal{B}_8 &= \{st^{-1}r^{-1}tsr, st^{-1}rtsr\} & \mathcal{C}_5 &= \{ts^{-1}t^{-2}r, st^{-1}r^{-1}ts, st^{-1}rts, t^2st^{-1}r\} \\ \mathcal{B}_9 &= \{st^{-1}r^{-1}t^2s, st^{-1}rt^2s\} & \mathcal{C}_6 &= \{t^{-1}r^{-1}t^2, t^{-1}rt^2, ts^{-1}t^{-1}r, t^2sr\} \\ \mathcal{B}_{10} &= \{t^{-1}r^{-1}tsrt, t^{-1}rtsrt\} & \mathcal{C}_7 &= \{ts^{-1}t^{-2}s, st^{-1}r^{-1}t^2, st^{-1}rt^2, t^2st^{-1}s\} \\ \mathcal{B}_{11} &= \{tsrts, s^{-1}t^{-1}rts\} & \mathcal{C}_8 &= \{st^{-1}r^{-1}t, st^{-1}rt, t^{-1}r^{-1}ts^{-1}, t^{-1}rts^{-1}\} \\ \mathcal{B}_{12} &= \{t^2srt, ts^{-1}t^{-1}rt\} & \mathcal{C}_9 &= \{t^{-1}r^{-1}ts, s^{-1}t^{-1}rt, t^{-1}rts, tsrt\} \\ \mathcal{B}_{13} &= \{st^{-1}rts^{-1}, st^{-1}r^{-1}ts^{-1}\} & \mathcal{C}_{10} &= \{tst^{-1}r, r^{-1}t^2s, rt^2s, s^{-1}t^{-2}r\} \\ \mathcal{B}_{14} &= \{t^2st^{-1}, ts^{-1}t^{-2}\} & & \end{aligned}$$

$$\begin{aligned}
\mathcal{D}_1 &= \{t, t^{-1}, rs, s^{-1}r^{-1}, t^{-2}, t^2, s^{-1}r^{-1}t, t^{-1}rs\} \\
\mathcal{D}_2 &= \{sr, r^{-1}s^{-1}, s^{-1}t, t^{-1}s, r^{-1}ts^{-1}, rts^{-1}, st^{-1}r^{-1}, st^{-1}r\} \\
\mathcal{D}_3 &= \{srs, st^{-1}, ts^{-1}, t^2r, s^{-1}t^{-2}s, s^{-1}t^2s, srt, tst^{-1}s\} \\
\mathcal{D}_4 &= \{tsr, r^{-1}ts, rts, s^{-1}t^{-1}r, r^{-1}tsrts, (rts)^2, s^{-1}t^{-1}rtsr, (tsr)^2\} \\
\mathcal{D}_5 &= \{r^{-1}t^2, rt^2, t^{-2}r^{-1}, t^{-2}r, r^{-1}tsrt, rtsrt, t^{-1}r^{-1}tsr, t^{-1}rtsr\} \\
\mathcal{D}_6 &= \{srt, s^{-1}t^2, t^{-1}r^{-1}s^{-1}, t^{-2}s, s^{-1}t^2srt, (srt)^2, ts^{-1}t^{-1}rts, t^2srt\}.
\end{aligned}$$

Proposition 7.1. *Let BI_{120} the binary icosahedral group and $\mathcal{C}(\text{BI}_{120}, \Gamma)$ be a commuting graph on BI_{120} . Then we have*

$$G = \begin{cases} K_2 & , \text{when } \Gamma = \mathcal{B}_i \text{ for } i = 1, \dots, 16 \\ K_4 & , \text{when } \Gamma = \mathcal{C}_i \text{ for } i = 1, \dots, 10 \\ K_8 & , \text{when } \Gamma = \mathcal{D}_i \text{ for } i = 1, \dots, 6. \end{cases}$$

Since we obtain complete graphs, all the metric properties of G are obvious.

8. COMMUTING GRAPHS, SIMPLE GRAPHS AND DYNKIN DIAGRAMS

Let $M = (m_{ij})_{1 \leq i, j \leq n}$ be an $n \times n$ symmetric matrix with $m_{ij} \in \mathbb{N} \cup \{\infty\}$ such that $m_{ii} = 1$ and $m_{ij} \geq 2$ if $i \neq j$. The Coxeter group of type M is defined as

$$W_M = \langle s_1, \dots, s_n \mid (s_i s_j)^{m_{ij}} = 1, 1 \leq i, j \leq n, m_{ij} < \infty \rangle.$$

See [11] for a standard reference. Observe that condition $m_{ij} = 1$ implies that $s_i^2 = 1$ for all i . The notation $m_{ij} = \infty$ is reserved for the case when there is no relation of the form $(s_i s_j)^m = 1$ for any m . For example, if $n = 1$ then $W_M = C_2$ the cyclic group of order 2 and if $n = 2$ then $M = \begin{pmatrix} 1 & m \\ m & 1 \end{pmatrix}$ and $W_M = D_{2m}$ is the dihedral group of order $2m$.

The following result observes that it is possible to realise any simple graph, i.e. any unweighted, undirected and containing no loops or multiple edges, can be realised as the commuting graph of an (in general) infinite Coxeter group.

Theorem 8.1. *Let G be a simple graph and denote by $\{1, \dots, n\}$ the vertex set. Then $G = \mathcal{C}(W_M, \Gamma)$ where $\Gamma = \{s_1, \dots, s_n\}$ is the set of generators of W_M and*

$$m_{ij} = \begin{cases} 1 & , \text{if } i = j \\ 2 & , \text{if } i \sim j \\ \geq 3 & , \text{if } i \not\sim j \end{cases}$$

Proof. In a Coxeter group W_M any pair of generators s_i and s_j commute if and only if $m_{ij} = 2$. Therefore, the commuting graph $\mathcal{C}(W_M, \Gamma)$ is defined as the graph with vertex set Γ and where two nodes s_i and s_j are joined if and only if $m_{ij} = 2$. This fact allows us to G as the commuting graph of a Coxeter group W_M with matrix type M with m_{ij} as stated in the result. \square

As an application we recover the simple laced Dynkin diagrams (as graphs), or the so called ADE classification, as commuting graphs of Coxeter groups:

Corollary 8.2. *Let $\Gamma = \{s_1, \dots, s_n\} \subset W_M$. Then every ADE Dynkin diagram is the commuting graph $\mathcal{C}(W_M, \Gamma)$ of a Coxeter group W_M .*

In Figure 1 we show every ADE Dynkin graph with the corresponding Coxeter matrix M , where the entries m_{ij} which are not shown consists of integers greater than 3 or ∞ .

C_n	BD_{4n}	BT_{24}	BO_{48}	BI_{120}
A_n	D_{n+2}	E_6	E_7	E_8
$\begin{pmatrix} 1 & 2 & & & \\ 2 & 1 & 2 & & \\ & 2 & 1 & 2 & \\ & & \ddots & \ddots & \ddots \\ & & & 2 & 1 & 2 \\ & & & & 2 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & & & & \\ 2 & 1 & 2 & & & \\ & 2 & 1 & 2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 2 & 1 & 2 & 2 \\ & & & & 2 & 1 & 2 \\ & & & & & 2 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & & & & \\ 2 & 1 & 2 & & & \\ & 2 & 1 & 2 & & \\ & & 2 & 1 & 2 & \\ & & & 2 & 1 & 2 \\ & & & & 1 & 2 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & & & & & \\ 2 & 1 & 2 & & & & \\ & 2 & 1 & 2 & & & \\ & & 2 & 1 & 2 & & \\ & & & 2 & 1 & 2 & \\ & & & & 2 & 1 & 2 \\ & & & & & 2 & 1 & 2 \\ & & & & & & 2 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & & & & & \\ 2 & 1 & 2 & & & & \\ & 2 & 1 & 2 & & & \\ & & 2 & 1 & 2 & & \\ & & & 2 & 1 & 2 & \\ & & & & 2 & 1 & 2 \\ & & & & & 2 & 1 & 2 \\ & & & & & & 2 & 1 & 2 \\ & & & & & & & 2 & 1 \end{pmatrix}$

FIGURE 1. Subgroups of $SL(2, \mathbb{C})$, ADE Dynkin diagrams G and the corresponding types of Coxeter groups for which $G = \mathcal{C}(W_M, \Gamma)$. The elements not shown in the matrices are numbers bigger or equal than 3.

Remark 8.3. Classically, given a Coxeter group W_M the so called Coxeter diagram is the graph with vertex set Γ and where two nodes i and j are joined if and only if $m_{ij} \geq 3$ (in the case $m_{ij} > 3$ the edge $i - j$ is labeled by m_{ij}). It is well known that finite and irreducible Coxeter groups are classified by diagrams of types $A_n (n \geq 1)$, $B_n (n \geq 3)$, $D_n (n \geq 2)$, E_6 , E_7 , E_8 , F_4 , G_2 , H_3 , H_4 and $I_2(m)$. It follows that we can recover Dynkin diagrams of ADE type as commuting graphs just by interchanging 2's and 3's in the matrices defining finite Coxeter groups of ADE type.

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